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WEAK AND STRONG CONVERGENCE THEOREMS FOR UNIFORMLY ASYMPTOTICALLY REGULAR NONEXPANSIVE SEMIGROUPS

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T:C\to C$ is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for all $x,y\in C$. We denote by $F(T)$ the set of fixed points of T. We know iteration procedures for finding a fixed point of a mapping T: Let x be an element of C and for each t with $0<t<1$, let x_t be a unique element of C satisfying $x_t=tx+(1-t)Tx_t$. In 1967, Browder [7] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Let x be an element of C and for each t with $0<t<1$, let x_t be a unique element of C satisfying*

$$x_t=tx+(1-t)Tx_t.$$

Then, $\{x_t\}$ converges strongly to the elemgent of $F(T)$ nearest to x as $t\downarrow 0$.

Reich [13] and Takahashi and Ueda [24] extended Browder's result to those of a Banach space. Using the idea of Shimizu and Takahashi [14, 15] and the notion of sequence of means, Shioji and Takahashi [16] proved the strong convergence of Browder's type sequences for nonexpansive semigroups (see also [17, 18, 19]). Recently, Domingues Benavides, Acedo and Xu [9] proved Browder's type strong convergence theorems for uniformly asymptotically regular one-parameter nonexpansive semigroups. Acedo and Suzuki [1] generalized Domingues Benavides, Acedo and Xu's results concerning the conditions of the sequences in real numbers.

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On the other hand, Xu and Ori [25] studied the following implicit iterative process for finite nonexpansive mappings $T_1, T_2, \ldots, T_r$ in a Hilbert space: $x_0 = x \in C$ and

$$x_n = \alpha_n + x_{n-1} + (1 - \alpha_n) T_n x_n \tag{1}$$

for every $n = 1, 2, \ldots$, where α_n is a sequence in $(0, 1)$ and $T_n = T_{n+r}$. And they proved the weak convergence of the iterative process defined by (1) in a Hilbert space. Motivated by [25], author and Takahashi [6] introduced an implicit iterative process for a nonexpansive semigroup and then prove a weak convergence theorem for the nonexpansive semigroup by using the idea of mean (see also [2, 3, 4]).

In this paper, we study the implicit iterations (1) for one-parameter nonexpansive semigroups and prove a weak convergence theorem for a uniformly asymptotically regular one-parameter nonexpansive semigroup in a Hilbert space. We also prove a weak convergence theorem for a uniformly asymptotically regular nonexpansive semigroup (see also [22, 23]). Further, we study Browder's type iterations for nonexpansive semigroups. Then, we prove strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces by using the idea of [1, 7, 9, 22, 23]. And we give a strong convergence theorem for the nonexpansive semigroup by the viscosity approximation method.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Then, for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [23] for more details. The following result is well-known; see also [23].

Lemma 2.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Let S be a semitopological semigroup. A semitopological semigroup S is called right (resp. left) reversible if any two closed left (resp. right) ideals of S have nonvoid intersection. If S is right reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $s \leq t$ if and only if $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{S+t}$, $s, t \in S$, where $\overline{A}$ is the closure of A. A commutative semigroup S is a directed system when the binary relation is defined by $s \leq t$ if and only if $\{s\} \cup (S+s) \supset \{t\} \cup (S+t)$.

Let C be a nonempty closed convex subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$.

We say that a Banach space E satisfies *Opial's condition* [12] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{2}$$

for each $y \in E$ with $y \neq x$. In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [10]. It is well known that this condition is equivalent to the analogous condition of $\overline{\lim}$ (see [5]). It is well known that Hilbert spaces satisfy Opial's condition (see [12, 23]).

Proposition 2.2 ([12]). *Let H be a Hilbert space. Let $\{x_n\}$ be a sequence in H converging weakly to $x \in H$. Then,*

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{3}$$

for each $y \in E$ with $y \neq x$.

3. Convergence theorems for one-paramater nonexpansive semigroups

In this section, we prove a weak convergence theorem for an asymptotically regular one-parameter nonexpansive semigroup by using the idea of [1, 9, 22, 23, 25]. Let C be a nonempty closed convex subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mapinngs of C into itself satisfying the following conditions is said to be one-parameter nonexpansive semigroup on C:

(i) for each $t \in \mathbb{R}^+$, $T(t)$ is nonexpansive;
(ii) $T(0) = I$;

(iii) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

We say that one-parameter nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ is asymptotically regular if

$$\lim_{s\to\infty} \|T(h+s)x - T(s)x\| = 0$$

for all $h \in \mathbb{R}^+$ and $x \in C$ (see also [22, 23]). The following lemma proved by Acedo and Suzuki ([1]).

Lemma 3.1 ([1]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{S} = \{T(s) : s \in \mathbb{R}^+\}$ be a one-paramater nonexpansive semigroup on C. Assume that $\mathcal{S} = \{T(s) : s \in \mathbb{R}^+\}$ is asymptotically regular, that is,*

$$\lim_{t\to\infty} \|T(h+t)x - T(t)x\| = 0$$

for all $h \in \mathbb{R}^+$ and $x \in C$. Then,

$$F(T(h)) = F(\mathcal{S})$$

for each $h \in \mathbb{R}^+$.

We say that one-paramater nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ is uniformly asymptotically regular if for every $h \in \mathbb{R}^+$ and for every bounded subset K of C,

$$\lim_{s\in\mathbb{R}^+} \sup_{x\in K} \|T(h+s)x - T(s)x\| = 0.$$

holds.

We prove a weak convergence theorem for a uniformly asymptotically regular one-parameter nonexpansive semigroup (see [1, 9]).

Theorem 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{S} = \{T(s) : s \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parametr nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)(T(m_n))x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{S}$.

4. Weak convergence theorems for nonexpansive semigroups

In this section, we prove a weak convergence theorem for an asymptotically regular nonexpansive semigroup by using the idea of [1, 9, 22, 23, 25]. Let C be a nonempty closed convex subset of a Hilbert space H, let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. We say that nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular if

$$\lim_{s \in S} \|T(h)T(s)x - T(s)x\| = 0$$

for all $h \in S$ and $x \in C$ (see also [22, 23]). The following lemma plays an important role in the proof of main theorem (see [1]).

Lemma 4.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Assume that $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular, that is,*

$$\lim_{t \in S} \|T(h)T(t)x - T(t)x\| = 0$$

for all $h \in S$ and $x \in C$. Then,

$$F(T(h)) = F(\mathcal{S})$$

for each $h \in S$.

We say that nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is uniformly asymptotically regular if for every $h \in S$ and for every bounded subset K of C,

$$\lim_{s \in S} \sup_{x \in K} \|T(h)T(s)x - T(s)x\| = 0.$$

holds.

We prove a weak convergence theorem for a uniformly asymptotically regular nonexpansive semigroup (see also [1, 9]).

Theorem 4.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{S}$.

5. Strong convergence theorems

Motivated by [1, 7, 9], we study Browder's type strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups (see also [22, 23]).

Theorem 5.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to Pu, where P is the metric projection from C onto $F(\mathcal{S})$.

We know that $f : C \to C$ is said to be a contraction on C if there exists $r \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq r\|x - y\|$$

for each $x, y \in C$. Using [21] and Theorem 5.1, we obtain the following strong convergence theorem by the viscosity approximation methods (see also [11]).

Theorem 5.2. *Let C be a nonempty closed convex subset of a Hilbert space H, let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : T \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{m_n\}$ be a sequence in $\mathbb{N}$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to Pu, where P is the metric projection from C onto $F(\mathcal{S})$.

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